Supplementary Information to "Intrinsic curvature determines the crinkled edges of "crenellated disks"."

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A. Total free energy

For a single surface of revolution terminated by two space curves, in terms of the angles α, β_i, and ϕ_i, and the differential geometric quantities defined in the main text, the total free energy is given by

$$F = F_H + F_{edge_1} + F_{edge_2} + F_{\lambda_1} + F_{\lambda_2}\,, \tag{1}$$

where

$$\begin{aligned} F_H = &\int_\epsilon^R dr(\pi+\phi_1+\phi_2)\left[\frac{\sigma r}{\cos\alpha} + \frac{k}{2r}\sin\alpha\tan\alpha \right. \\ &\left. + \frac{k}{2} r\cos\alpha\left(\frac{d\alpha}{dr}\right)^2\right] \\ &+ (k+\bar{k})\int_\epsilon^R \frac{dr}{r}\left(\tan\beta_1+\tan\beta_2\right) \\ &- (k+\bar{k})\cos\alpha(R)(\pi+\phi_1(R)+\phi_2(R)) \\ &+ (k+\bar{k})\cos\alpha(\epsilon)(\pi+\phi_1(\epsilon)+\phi_2(\epsilon)) \end{aligned} \tag{2}$$

$$\begin{aligned} F_{edge_i} = \int_\epsilon^R dr &\left[\frac{\gamma_i}{\cos\alpha\cos\beta_i} + k_b\frac{\kappa_{g,i}^2+\kappa_{n,i}^2}{\cos\alpha\cos\beta_i}\right. \\ &\left. - h_i\frac{\sin\beta_i}{\cos\alpha}\left(\kappa_2-\kappa_1\right)\right]\,, \end{aligned} \tag{3}$$

$$F_{\lambda_i} = \int_\epsilon^R dr\lambda_i\left(r\cos\alpha\frac{d\phi_i}{dr} - \tan\beta_i\right)\,. \tag{4}$$

The index of the edges is $i = \{1,2\}$, ϵ is the cut-off length, which is given by $\epsilon \equiv -R\sin\alpha(R)$ for the catenoid.

B. Euler-Lagrange equations

The variations $\frac{\delta F}{\delta g_j}$, where $g_j = \{\alpha, \beta_1, \beta_2, \phi_1, \phi_2\}$, are equivalent to the Euler-Lagrange (EL) equations. For one independent and several dependent variables, the EL equations are given by [1]

$$\frac{\partial f}{\partial g_j} = \frac{d}{dx}\frac{\partial f}{\partial g_j'} \tag{5}$$

where $f = f(r, g_j, g_j')$ is the free energy density given by $F = \int dr f(r, g_j, g_j')$. Primes denote derivatives $\frac{dg_j}{dr}$. The set of EL Equations in Eq. (5) contains ten coupled nonlinear first-order differential equations. Evaluating Eq. (5), these equations are calculated as

$$\begin{aligned} u_1 \equiv \frac{\partial f}{\partial \alpha'} =&\pi k r\cos\alpha\frac{d\alpha}{dr} + 2k_b\kappa_{n,1}\cos\beta_1 + 2k_b\kappa_{n,2}\cos\beta_2 \\ &+ h_1\sin\beta_1 + h_2\sin\beta_2\,, \end{aligned} \tag{6}$$

$$u_2 \equiv \frac{\partial f}{\partial \beta_1'} = 2k_b\kappa_{g,1}\,, \quad u_3 \equiv \frac{\partial f}{\partial \beta_2'} = 2k_b\kappa_{g,2}\,, \tag{7}$$

$$u_4 \equiv \frac{\partial f}{\partial \phi_1'} = \lambda_1 r\cos\alpha\,, \quad u_5 \equiv \frac{\partial f}{\partial \phi_2'} = \lambda_2 r\cos\alpha\,, \tag{8}$$

$$\begin{aligned} u_1'\cos^2\alpha = &\sigma\pi r\sin\alpha + \frac{k\pi\sin\alpha}{2}\left[\frac{2}{r} - \frac{\sin^2\alpha}{r} - r\cos^2\alpha\left(\frac{d\alpha}{dr}\right)^2\right] \\ &+ \sin\alpha\sec\beta_1(\gamma_1 + k_b\kappa_{n,1}^2 - k_b\kappa_{g,1}^2) \\ &+ \sin\alpha\sec\beta_2(\gamma_2 + k_b\kappa_{n,2}^2 - k_b\kappa_{g,2}^2) \\ &+ 2k_b\kappa_{n,1}\sec\beta_1\left(\frac{\cos^2\alpha\sin^2\beta_1}{r} - \frac{\sin 2\alpha}{2}\frac{d\alpha}{dr}\cos^2\beta_1\right) \\ &+ 2k_b\kappa_{n,2}\sec\beta_2\left(\frac{\cos^2\alpha\sin^2\beta_2}{r} - \frac{\sin 2\alpha}{2}\frac{d\alpha}{dr}\cos^2\beta_2\right) \\ &- \frac{1}{r}\left(h_1\sin\beta_1 + h_2\sin\beta_2\right) \\ &- \frac{\sin 2\alpha}{2}(\lambda_1\tan\beta_1 + \lambda_2\tan\beta_2)\,, \end{aligned} \tag{9}$$

$$\begin{aligned} \cos\alpha\cos^2\beta_1 u_2' = &(k+\bar{k})\frac{\cos\alpha}{r} + \gamma_1\sin\beta_1 - \lambda_1\cos\alpha \\ &+ k_b\sin\beta_1(\kappa_{n,1}^2 - \kappa_{g,1}^2) + \frac{2k_b}{r}\kappa_{g,1}\cos\alpha \\ &+ \left(4k_b\kappa_{n,1}\cos\beta_1 - h_1\cos\beta_i\cot\beta_1\right)\tau_{g,1}\,, \end{aligned} \tag{10}$$

$$\cos\alpha\cos^2\beta_2 u_3' = (k+\bar{k})\frac{\cos\alpha}{r} + \gamma_2\sin\beta_2 - \lambda_2\cos\alpha + k_b\sin\beta_2(\kappa_{n,2}^2 - \kappa_{g,2}^2) + \frac{2k_b}{r}\kappa_{g,2}\cos\alpha + (4k_b\kappa_{n,2}\cos\beta_2 - h_2\cos\beta_2\cot\beta_2)\,\tau_{g,2}\,, \tag{11}$$

$$u_4' = u_5' = \frac{\sigma r}{\cos\alpha} + \frac{k}{2r}\sin\alpha\tan\alpha + \frac{k}{2}r\cos\alpha\left(\frac{d\alpha}{dr}\right)^2 . \tag{12}$$

In Eq. (9), the two constraints $r\cos\alpha\frac{d\phi_1}{dr} = \tan\beta_i$ are used. Based on Eqs. (8) and (12), we conclude that $\lambda \equiv \lambda_1 = \lambda_2$. Then, the nine unknowns to be solved are $\alpha, \alpha', \beta_1, \beta_1', \beta_2, \beta_2', \phi_1, \phi_2, \lambda$.

Note that we neglect the contribution of the variations $\frac{\delta F}{\delta\phi_i}$ to the surface and edge deformations, as explained in the main text. Therefore, the structure in our approximation is governed solely by $\alpha, \alpha', \beta_1, \beta_1', \beta_2, \beta_2'$. In the case of the minimal surface, since α and α' are calculated analytically (see main text), the equations governing β_1, β_1' and β_2, β_2' are decoupled from each other.

C. Boundary conditions

The torque-free boundary conditions are given by Eqs. (6), (7), and (8) being equal to zero [1]. Sufficiently away from the rotation axis of the surface, in 2D membrane limit, $\alpha = 0$, $\alpha' = 0$, and $\alpha'' = 0$. Then, $\kappa_{n,i} = 0$. Plugging in these results to Eq. (9), we find that $\sin\beta_i = A_i/r$. This rather expected result applies to a straight line in polar coordinates, where the shortest distance between the origin and the straight line is A_i (or, the in-plane protrusion amplitude). When $h_i = 0$ as in the achiral limit, then the same relation still holds, and this time the torque-free condition $\frac{\partial f}{\partial\alpha'} = 0$ is satisfied. The geodesic curvature $\kappa_{g,i}$ of a straight line vanishes in 2D, hence $\frac{\partial f}{\partial\beta_i'} = 0$. In the 2D membrane limit λ must be zero. This is best seen when a curve with the free energy given in Eq. (3) is minimized on a plane. The resulting EL equations are independent of ϕ_i, then $\lambda = 0$. Hence $\frac{\partial f}{\partial\phi_i'} = 0$, away from the rotation axis of the nonplanar surface. Plugging in these results into Eqs. (10) and (11), we find that $k+\bar{k} = \gamma_1 A_1 = \gamma_2 A_2$. For a minimal surface, k does not contribute to this equation.

At the cusp, ideally, the boundary conditions are $\alpha = -\pi/2$ and $\beta_i = 0$ (again, we ignore $\frac{\delta F}{\delta\phi_i}$ dependence of the structure). In the case of the minimal surface, since $\epsilon \approx 10^{-2}R$, we take $\alpha = 0.95$ which nearly corresponds to a 70^o slope. This slope is steep enough as evidenced in Fig. 3(a) in the main text.

[1] Arfken G., Weber H. *Mathematical Methods for Physicists, Sixth Edition* (Elsevier Inc., 2005).