Supporting Information A

Single circular insertion in a folded membrane

The velocity field around a circular disc in a 2-dimensional fluid membrane of viscosity η that is supported on a flat substrate with friction coefficient β_{m-s} obeying (1) was found [24] to be

$$v_r = \cos\theta \left[\frac{a_1}{r^2} + \frac{b_1}{r\xi^2} K_1(\xi r)\right]$$

$$v_\theta = \sin\theta \left[\frac{a_1}{r^2} + \frac{b_1}{2\xi}\left(K_0(\xi r) + K_2(\xi r)\right)\right] \tag{A.1}$$

with the constants

$$a_1 = v_0 a^2 \frac{K_2(\xi a)}{K_0(\xi a)} \;\; ; \;\; b_1 = -\frac{2 v_0 \xi}{K_0(\xi a)}$$

where a is the radius of the disc, v_0 the velocity of the disc in direction $\theta = 0$, $\xi^2 = \beta_{m-s}/\eta$ and $K_n(x)$ is the modified Bessel function of the second kind. The force exerted on the membrane by the moving disc is found [24] from the stress tensor to be

$$\overrightarrow{f_{disk}} = \pi a^2 \beta_{m-s} \overrightarrow{v_0} \left(2\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right) \tag{A.2}$$

and the total force that the fluid exerts on the substrate is

$$\overrightarrow{f_{fric}} = \beta_{m-s} \int (v_r \cos\theta - v_\theta \sin\theta) r dr d\theta = \beta_{m-s} \pi a^2 \overrightarrow{v_0} \left(\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right). \tag{A.3}$$

We wish to integrate separately the velocity (A.1) in front and behind an imaginary line (the folding line) perpendicular to the direction of the velocity of the disc. The symmetry of the system allows us to integrate only the x-component of the velocity, since the y-component vanishes. We assume that the distance R between the disc and the folding line is large enough $\xi R \gg 1$, and separate the integration over the long (proportional to a_1) and short (proportional to b_1) terms of the velocity (A.1). Integration in front of the folding line will include integration

Electronic Supplementary Material (ESI) for Soft Matter

over the short term with infinite boundaries, plus integration over the long term on semi-infinite plain, with the folding line in a finite distance R. The short term integration is given by

$$\int \overrightarrow{v_{sr}} dA = \int (v_r \cos\theta - v_t \sin\theta)_{sr} r dr d\theta = \pi a^2 \overrightarrow{v_0} \left(\frac{K_2(\xi a)}{K_0(\xi a)} - 1 \right) \tag{A.4}$$

while the long term integration in front of the folding line is

$$\int \overrightarrow{v_{lr_v}} dA = \int \left(\frac{a_1}{r^2} \cos 2\theta \right) r dr d\theta = -\frac{\pi}{2} a^2 \overrightarrow{v_0} \frac{K_2(\xi a)}{K_0(\xi a)}. \tag{A.5}$$

Unexpectedly, the integration does not depend on the distance from the folding line. Integration behind the folding line yields

$$\int \overrightarrow{v_{lr_d}} dA = \frac{\pi}{2} a^2 \overrightarrow{v_0} \frac{K_2(\xi a)}{K_0(\xi a)} \tag{A.6}$$

and the sum of (A.5) and (A.6) vanishes as expected.

The force exerted by the substrate on the side of the membrane encircling the disc is now

$$\overrightarrow{f_{m-s}^{v}} = -\beta_{m-s} \int \left(\overrightarrow{v_{sr}} + \overrightarrow{v_{lr_v}} \right) dA = -\beta_{m-s} \pi a^2 \overrightarrow{v_0} \left(\frac{K_2(\xi a)}{2K_0(\xi a)} - 1 \right) \tag{A.7}$$

And at the other side

$$\overrightarrow{f_{m-s}^{d}} = -\beta_{m-s} \int \overrightarrow{v_{lr_d}} dA = -\beta_{m-s} \pi a^2 \overrightarrow{v_0} \frac{K_2(\xi a)}{2K_0(\xi a)} \tag{A.8}$$

Inserting (A.3), (A.7) and (A.8) into the steady state conditions of the movement of the membrane and of the filament (see main text (5-7)) in order to find the overall velocity, without neglecting term of order $\pi a^2 / A$ gives

$$\overrightarrow{v_m} = \frac{\overrightarrow{f_{pol}}}{2A\beta_{m-s}} \frac{\left(\frac{\beta_{a-s}}{\beta_{m-s}} - 1\right)\left(1 + \frac{\pi a^2}{2A}\left(\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right)\right)}{\left(2\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right)\left(1 - \frac{\pi a^2}{A}\right) + \frac{\beta_{a-s}}{\beta_{m-s}}\left(1 + \frac{\pi a^2}{A}\left(\frac{3}{2}\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right)\right)} \tag{A.9}$$

$$\overrightarrow{v_r} = \frac{-\overrightarrow{f_{pol}}}{2A\beta_{m-s}} \frac{\frac{K_2(\xi a)}{K_0(\xi a)} + \frac{\pi a^2}{2A}\left(\frac{\beta_{a-s}}{\beta_{m-s}} - 1\right)\left(\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right)}{\left(2\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right)\left(1 - \frac{\pi a^2}{A}\right) + \frac{\beta_{a-s}}{\beta_{m-s}}\left(1 + \frac{\pi a^2}{A}\left(\frac{3}{2}\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right)\right)} \tag{A.10}$$

$$\overrightarrow{v_{ret}} = \frac{-\overrightarrow{f_{pol}}}{\pi a^2\beta_{m-s}} \frac{1 + \frac{\pi a^2}{A}\frac{K_1(\xi a)}{\xi a K_0(\xi a)}}{\left(2\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right)\left(1 - \frac{\pi a^2}{A}\right) + \frac{\beta_{a-s}}{\beta_{m-s}}\left(1 + \frac{\pi a^2}{A}\left(\frac{3}{2}\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right)\right)} \tag{A.11}$$

Supporting Information B

Two insertions at finite distance

We now consider a similar system of 2-dimensional fluid that interacts through friction with a substrate. This time the fluid contains two discs with radii ρ_1, ρ_2 and a distance R from each other. Each of the discs moves with a different constant velocity v_1, v_2 and the only restriction is that they both move perpendicular to the edge of the membrane and parallel to each other. We let the angle between their common line and the edge of the membrane to be ϕ. This problem is not solvable analytically so we wish to find the leading correction term in the small parameter (ρ/R). We can look at the velocity field as the sum of two contributions, each from every disc, but the sum of all contributions must satisfy the boundary conditions on each of the discs edges. As seen in Appendix B, the velocity field is the sum of a long-ranged term (the $a_1$term) and a short-ranged term (b_1), so we expect that the correction to the single disc problem will contain only the long range term.

We mark the coordinates of disc #1 (r_1, θ_1) and similarly for disc #2. The leading term of the velocity induced by #2 in the vicinity of #1, i.e. where $r_2 \cong R, \theta_2 \cong -\frac{\pi}{2} - \phi$, and $\xi R \gg 1$, is $\vec{v} = \frac{a_2}{R^2}\left(\cos\theta_2\,\widehat{r_2} + \sin\theta_2\,\widehat{\theta_2}\right) = \frac{a_2}{R^2}(-\cos 2\phi\,\hat{x} + \sin 2\phi\,\hat{y})$. In term of the coordinates of disc #1 the velocity is approximated by

$$v_{r_1} \cong \cos\theta\left[\frac{a_1}{r_1^2}+\frac{b_1}{\xi^2 r_1}K_1(\xi r_1)-\frac{a_2}{R^2}\cos 2\phi\right]+\sin\theta\left[\frac{c_1}{r_1^2}+\frac{d_1}{\xi^2 r_1}K_1(\xi r_1)+\frac{a_2}{R^2}\sin 2\phi\right]$$

$$\begin{aligned} v_{\theta_1} &\cong \sin\theta\left[\frac{a_1}{r_1^2}+\frac{b_1}{2\xi}\big(K_0(\xi r_1)+K_2(\xi r_1)\big)+\frac{a_2}{R^2}\cos 2\phi\right] \\ &\quad+\cos\theta\left[-\frac{c_1}{r_1^2}-\frac{d_1}{2\xi}\big(K_0(\xi r_1)+K_2(\xi r_1)\big)+\frac{a_2}{R^2}\sin 2\phi\right] \end{aligned} \qquad \text{(B.1)}$$

where we keep additional terms in the solution of (A.1) that do not vanish in the leading correction. The no-slip boundary conditions on both edges now lead to

$$\begin{aligned} a_1 &= v_1\rho_1^2\frac{K_2(\xi\rho_1)}{K_0(\xi\rho_1)}+v_2\rho_1^2\frac{K_2(\xi\rho_1)}{K_0(\xi\rho_1)}\frac{K_2(\xi\rho_2)}{K_0(\xi\rho_2)}\cos 2\phi\left(\frac{\rho_2}{R}\right)^2+O\left(\frac{\rho_2^4}{R^4}\right) \\ b_1 &= -\frac{2v_1\xi}{K_0(\xi\rho_1)}-\frac{2v_2\xi}{K_0(\xi\rho_1)}\frac{K_2(\xi\rho_2)}{K_0(\xi\rho_2)}\cos 2\phi\left(\frac{\rho_2}{R}\right)^2+O\left(\frac{\rho_2^4}{R^4}\right) \\ c_1 &= -\rho_1^2 v_2\frac{K_2(\xi\rho_2)}{K_0(\xi\rho_2)}\frac{K_2(\xi\rho_1)}{K_0(\xi\rho_1)}\sin 2\phi\left(\frac{\rho_2}{R}\right)^2+O\left(\frac{\rho_2^4}{R^4}\right) \\ d_1 &= \frac{2\xi v_2}{K_0(\xi\rho_1)}\frac{K_2(\xi\rho_2)}{K_0(\xi\rho_2)}\sin 2\phi\left(\frac{\rho_2}{R}\right)^2+O\left(\frac{\rho_2^4}{R^4}\right). \end{aligned} \qquad \text{(B.2)}$$

A similar result for a_2, b_2, c_2, d_2 is obtained be replacing the subscripts $1 \leftrightarrow 2$.

The force exerted by disc #1 in the x-direction is now

$$\begin{aligned} f_{disk} &= -\rho_1\int_0^{2\pi}(\sigma_{rr}\cos\theta-\sigma_{r\theta}\sin\theta)d\theta \\ &= \pi\rho_1^2\beta_{m-s}\overrightarrow{v_1}\left(2\frac{K_2(\xi\rho_1)}{K_0(\xi\rho_1)}-1\right)+2\pi\rho_1^2\beta_{m-s}\overrightarrow{v_2}\frac{K_2(\xi\rho_1)}{K_0(\xi\rho_1)}\frac{K_2(\xi\rho_2)}{K_0(\xi\rho_2)}\cos 2\phi\left(\frac{\rho_2}{R}\right)^2 \end{aligned} \qquad \text{(B.3)}$$

Two discs anchored symmetrically in opposite membranes

We wish to examine the special case of two identical discs with radius a, a distance R between them, moving in a homogeneous fluid away from each other with opposite velocities ($\overrightarrow{v_1} = -\overrightarrow{v_2} \equiv \vec{v}, \phi = \pi/2$). In this case the velocity at the vicinity of each disc, where $r \ll R$ (with the origin at the center of the disc) is

$$v_r \cong \cos\theta\left[a_1\left(\frac{1}{r^2}-\frac{1}{R^2}\right)+\frac{b_1}{\xi^2 r}K_1(\xi r)\right]$$

$$v_\theta \cong \sin\theta\left[a_1\left(\frac{1}{r^2}+\frac{1}{R^2}\right)+\frac{b_1}{2\xi}\left(K_0(\xi r)+K_2(\xi r)\right)\right]$$

where the no-slip condition on the boundary of each disc gives

$$a_1 \cong va^2\frac{K_2(\xi a)}{K_0(\xi a)}\left(1+\frac{a^2}{R^2}\frac{K_2(\xi a)}{K_0(\xi a)}\right)$$

$$b_1 \cong -\frac{2v\xi}{K_0(\xi a)}\left(1+\frac{a^2}{R^2}\frac{K_2(\xi a)}{K_0(\xi a)}\right),$$

the force that the fluid exerts on each disc is by (B.3):

$$\overrightarrow{f_{disk}} = -\pi a^2\beta_{m-s}\vec{v}\left(2\frac{K_2(\xi a)}{K_0(\xi a)}-1+2\left(\frac{K_2(\xi a)}{K_0(\xi a)}\right)^2\frac{a^2}{R^2}\right). \tag{B.4}$$

and the friction force that each half of the membrane exerts on the substrate is simply

$$\overrightarrow{f_{fric}} = \beta_{m-s}\int \vec{v}dA \cong -\pi a^2\beta_{m-s}\vec{v}. \tag{B.5}$$

In order to verify these results, we wish to calculate the total force on the line of equal distance between the discs f_{fold}. If we calculated correctly, the total force on the each half of the membrane should vanish: $f_{fold}+f_{fric}+f_{disk}=0$.

In calculating the force exerted on all of the outer shell of each side we assume $\xi R \gg 1$, so we only need to keep the long range terms of each disc. The resulting velocity field, in Cartesian coordinates with the origin at the middle between the two discs, is

$$v_x \cong a_1\left[\frac{(x-R/2)^2-y^2}{((x-R/2)^2+y^2)^2}-\frac{(x+R/2)^2-y^2}{((x+R/2)^2+y^2)^2}\right]$$

$$v_y \cong 2a_1\left[\frac{(x-R/2)y}{((x-R/2)^2+y^2)^2}-\frac{(x+R/2)y}{((x+R/2)^2+y^2)^2}\right].$$

Notice how v_x vanishes at points of equal distance between the two discs ($x=0$). This justify our choice for the configuration of the original problem: the velocity field in the presence of two discs is similar (up to the second order in ρ/R) to the field in the presence of a single disc and a straight edge, a distance R appart. On the shell at infinity, the y-component of the force cancels due to the symmetry of the problem, while the x-component vanishes due to the rapid decline in the velocity.

On the line between the two discs ($x = 0$), on the other hand, the force does not vanish:

$$\overrightarrow{f_{fold}} = \int_{-\infty}^{\infty} \sigma_{xx}|_{x=0} dy = \int_{-\infty}^{\infty} \left(\tau + 2\eta \frac{\partial v_x}{\partial x}\right)_{x=0} dy$$
$$= 2\beta\pi\vec{v}a^2 \frac{K_2(\xi a)}{K_0(\xi a)}\left(1 + \frac{a^2}{R^2}\frac{K_2(\xi a)}{K_0(\xi a)}\right). \tag{B.6}$$

As expected,

$$\overrightarrow{f_{fold}} + \overrightarrow{f_{fric}} + \overrightarrow{f_{disk}} = 0. \tag{B.7}$$

Notice again that $\overrightarrow{f_{fold}} + \overrightarrow{f_{disk}} = \pi a^2 \beta_{m-s}\vec{v} > 0$, which means that if we consider only half-infinite plane and apply a certain force that causes the disc to move, the resulting reaction force on the shell at the finite distance is larger than the force we initially applied, independent on the distance from the disc.

Two discs anchored in the same side of the membrane

In exactly the same manner as in the last two subcases, we shall now work out the example of two identical discs moving in the ventral membrane with common velocity $\vec{v}$, distant R appart. In this case (B.3) becomes

$$f_{disk} = \pi a^2 \beta_{m-s}\vec{v}\left[\left(2\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right) + 2\left(\frac{K_2(\xi a)}{K_0(\xi a)}\right)^2 \cos 2\phi \frac{a^2}{R^2}\right] \tag{B.8}$$

the integrated velocity in the ventral membrane becomes

$$\int \overrightarrow{v_v} dA \cong -\pi(b_1 + b_2)\frac{aK_1(\xi a)}{\xi^2} - \frac{\pi}{2}(a_1 + a_2) - \frac{\pi a^2}{R^2}\cos 2\phi\,(a_1 + a_2)$$
$$\cong -\pi a^2 \vec{v}\left[2 - \frac{K_2(\xi a)}{K_0(\xi a)} - \left(\frac{K_2(\xi a)}{K_0(\xi a)}\right)^2 \cos 2\phi \frac{a^2}{R^2}\right] \tag{B.9}$$

While the integrated velocity in the dorsal membrane is simply

$$\int \overrightarrow{v_d} dA \cong \frac{\pi}{2}(a_1 + a_2) \cong \pi a^2 \vec{v} \frac{K_2(\xi a)}{K_0(\xi a)} \left[1 + \frac{K_2(\xi a)}{K_0(\xi a)} \cos 2\phi \frac{a^2}{R^2}\right]. \tag{B.10}$$

The total friction force exerted by the substrate would just be (B.9) or (B.10) multiplied by $-\beta_{m-s}$.

Supporting Information C

Constant density of inclusions

We now generalize the previous case and consider a membrane with multiple inclusions. For simplicity, we consider all inclusions to have the same radius a, all inclusions in the ventral side have common velocity $\overrightarrow{v_v}$, and all inclusions in the dorsal membrane have common velocity $\overrightarrow{v_d}$. We use the mean field approach and assume all inclusions are evenly distributed in the membrane, so we consider the membrane to have constant surface density of inclusions σ.

To this end we first calculate the force exerted on the membrane by one rod provided that similar rods move in the membrane, each characterized by velocity $\overrightarrow{v_i}$ and distance $\overrightarrow{R_i} = (R_i, \phi_i)$ relative to the rod under consideration. Following the same steps that led to (B.3), we conclude that the velocity field in the vicinity of a specific inclusion in the ventral membrane would include (long term) contributions from all other inclusions

$$\overrightarrow{f_{disc_v}} = \pi a^2 \beta_{m-s} \overrightarrow{v_v} \left(2\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right) - 2\pi a^2 \beta_{m-s} \left(\frac{K_2(\xi a)}{K_0(\xi a)}\right)^2 \sum_i \overrightarrow{v_i} \cos 2\phi_i \left(\frac{a}{R_i}\right)^2, \tag{C.1}$$

while for inclusions in the dorsal membrane we replace the subscripts $v \leftrightarrow d$ in (C.1). Replacing the summation in (C.1) by integration over the membrane area, we obtain

$$\begin{aligned} \overrightarrow{f_{disc_v}} &= \pi a^2 \beta_{m-s} \left[v_v \left(2\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right) + \left(\frac{K_2(\xi a)}{K_0(\xi a)}\right)^2 \sigma(\overrightarrow{v_v} - \overrightarrow{v_d}) \right] \\ \overrightarrow{f_{disc_d}} &= \pi a^2 \beta_{m-s} \left[v_d \left(2\frac{K_2(\xi a)}{K_0(\xi a)} - 1\right) + \left(\frac{K_2(\xi a)}{K_0(\xi a)}\right)^2 \sigma(\overrightarrow{v_d} - \overrightarrow{v_v}) \right] \end{aligned} \tag{C.2}$$

And the total force that all rods exerts on the membrane would be the above force times the number of rods $\frac{\sigma A}{\pi a^2}$.