Electronic Supplementary Material (ESI) for Soft Matter

Curling and rolling dynamics of naturally curved ribbons (ESI)

Octavio Albarrán Arriagada,[a] Gladys Massiera,[a] and Manouk Abkarian,*[a]

1 The critical natural radius a_0^*

1.1 The Heavy Elastica Equation

We are interested to find the differential equation that describes the shape of a ribbon that bends with planar deformations under its own weight. In Fig.1, a schematic of the problem is presented: one end of the ribbon is immobilized by fixing its local tangential vector and the other extremity is left free. We define the natural radius of curvature a_0, the tangential angle θ and the arc length position S, which runs from zero to the full length S_β of the material.

For static equilibrium, the equations of force and torque are given by:

$$\partial_S \mathbf{F} + \mathbf{K} = 0 \tag{1}$$

and

$$\partial_S \mathbf{M} + \mathbf{t} \times \mathbf{F} = 0 \tag{2}$$

where $\mathbf{K}$ is the external force per unit of area; $\mathbf{F}$ is the internal force resultant on the cross-section and $\mathbf{M}$ is the torque resultant per unit length. For planar deformations, the torques are connected with the curvature by: $\mathbf{M} = -B(\kappa_0 - \kappa)\mathbf{e}_3$, where B is the bending stiffness, $\kappa_0 = 1/a_0$ is the natural curvature and $\kappa = \partial_S \theta = \theta'$ is the local curvature.

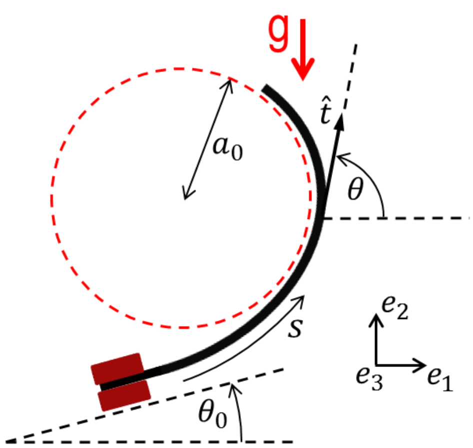

Fig. 1 Scheme of the static configuration of a ribbon of natural radius a_0 that is deformed by its own weigth.

Solving Eq.1 for the gravitational interaction $\mathbf{K} = -g\sigma\mathbf{e}_2$ (g is the gravitational aceleration and σ the surface density) we get $\mathbf{F} = g\sigma(S - S_\beta)\mathbf{e}_2$, and then, by Eq.2, the general equation for static equilibrium is found:

$$\theta'' + \frac{g\sigma}{B}(S - S_\beta)\cos\theta = 0 \tag{3}$$

To nondimensionalize, we introduce the parameter $\chi = S/S_\beta$, then eq.3 becomes

$$\frac{d^2\theta(\chi)}{d\chi^2} - \frac{(1-\chi)}{b}\cos\theta(\chi) = 0 \tag{4}$$

where $b = \frac{B}{\sigma g S_\beta^3}$. Because of the origin of the problem, this equation must be subjected to the boundary conditions: $\theta(0) = \theta_0$ and $\frac{d\theta}{d\chi}(1) = \frac{S_\beta}{a_0}$.

1.2 Numerical solution

When the angle $\theta(\chi)$ and its derivative $\frac{d\theta(\chi)}{d\chi}$ in $\chi = 0$ are imposed, using finite differences method we can easily solve the Eq.4 numerically. First we aproximate the second derivative,

$$\frac{d^2\theta}{d\chi^2} \approx \frac{\theta(\chi + \Delta\chi) - 2\theta(\chi) + \theta(\chi - \Delta\chi)}{\Delta\chi^2}$$

and discretize the domain of the solution: $\chi \to \chi_n = n\Delta\chi$. Now, considering $\theta(\chi) \to \theta_n = \theta(\chi_n)$ and $\Delta\chi = 1/N$ (where N is the number of intervals of the domain), we get the following recursive formula

$$\theta_{n+1} = \frac{1}{b}\left(\frac{1}{N^2} - \frac{n}{N^3}\right)\cos\theta_n + 2\theta_n - \theta_{n-1} \tag{5}$$

where $n = 1, 2, 3, ..., N-1$. In order to find the complete numerical solution, we start with the initial values, θ_0 and θ_1, that are given by the boundary conditions of the problem.

For the problem of the equilibrium shape of the frustrated curling, the boundary conditions for rods are $\theta(0) = \theta_0 = 0$ and $\frac{d\theta(0)}{d\chi} = 0 \Rightarrow \theta_1 = 0$. Thus, with the recursive formula, for each number b, we have access to the entire angular variation of the rod. Especially, at the free boundary the curvature is $\frac{d\theta(1)}{d\chi} = \frac{S_\beta}{a_0}$ (which is also the dimensionless natural curvature). In Fig.2 we have plotted the numerical solutions for the parameter $1/b$ associated with the normalized curvature S_β/a_0 (boundary condition at the free end), the graph shows that $1/b$ can not be higher than 45.63, otherwise S_β/a_0 becomes negative and the solution is not more compatible with the conditions of the problem.

[a] *Laboratoire Charles Coulomb, Université Montpellier 2 - CNRS, UMR 5221,34095 Cedex 5, Montpellier, France; E-mail: Manouk.Abkarian@univ-montp2.fr*

For ribbons, the problem is more subtle because the curvature at $\chi = 0$ is given by $\frac{d\theta(0)}{d\chi} = \nu \frac{S_\beta}{a_0}$ that must be also compatible with the boundary condition at the free end, which implies that θ_1, θ_{N-1} and θ_N are explicitly connected:

$$\theta_1 = (\theta_N - \theta_{N-1})\,\nu$$

Knowing ν, we run the iterative formula of Eq.5, performing a searching loop where, for a specific b, the initial estimate of θ_1 will be given by the dimensionless natural curvature of the associated rod solution. Then, writing the curvature at the free end as $\kappa_N(b, \theta_1)$, the first iterative solution will be written $\theta_1^{(1)} = \frac{\nu}{N}\kappa_N(b, \theta_1 = 0)$. Using this same idea we can produce a better estimate $\theta_1^{(2)} = \frac{\nu}{N}\kappa_N\left(b, \theta_1^{(1)}\right)$. Thus, $\theta_1^{(i)}$ can be improved for any required accuracy using the algorithm:

$$\theta_1^{(i+1)} = \frac{\nu}{N}\kappa_N\left(b, \theta_1^{(i)}\right)$$

For a ribbon of $\nu = 0.38$, the Fig.3 also shows the relation between $1/b$ and S_β/a_0 obtained by means of the numerical solution.

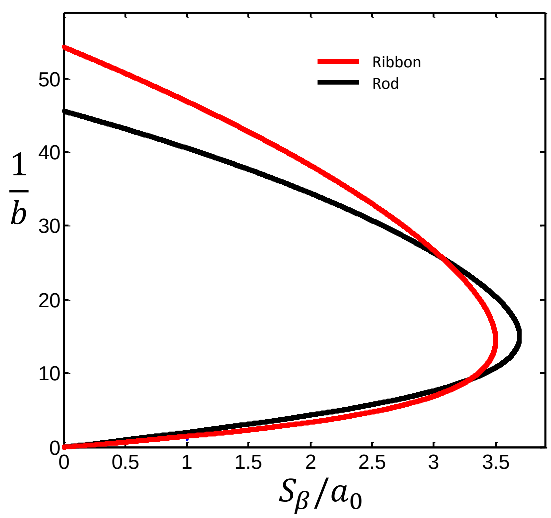

Fig. 2 Numerical Solution (with finite differences method) of the relation between the heavy-elastica constant $1/b$ and the associated boundary condition S_β/a_0. The red line is associated with a ribbon with $\nu = 0.38$.

1.3 The limit for static equilibrium

When a_0 is lower than the critical value a_0^*, the stored elastic energy of the ribbon is higher than its gravitational potential energy and curling starts. However, when $a_0 \gtrsim a_0^*$, the ribbon adopts a static configuration that we characterize by two variables: the height Y_β of the free end β of the ribbon, and the curvilinear length S_β between the contact with the substrate, α, and β (see picture in Fig.3B).

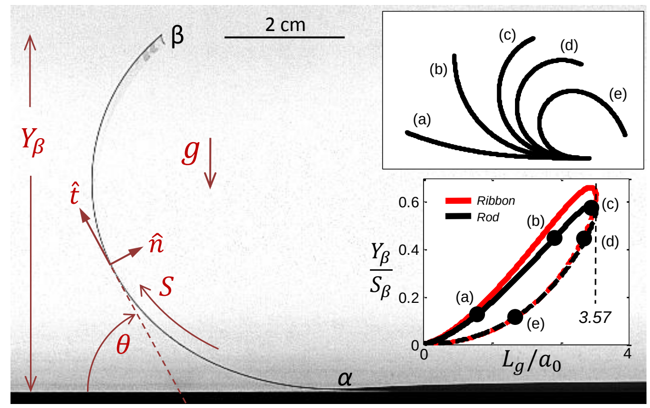

Fig. 3 Image of a ribbon in static equilibrium with gravity (PVC film 100 μm thick, $a_0 = 4.0 \pm 0.1$ cm and $W = 3.5$ cm). Upper inset: Numerical solutions for static equilibrium shapes obtained using Eq.4. Positions are normalized by S_β. Lower inset: Equilibrium diagram Y_β/S_β versus L_g/a_0 ($a_0 > 0.28L_g$). Two solutions are represented one stable, one unstable (dashed line) for each L_g/a_0 (red line obtained for a ribbon with $\nu = 0.4$). On the plot, letters indicate the shape obtained by the numerical solution.

To deduce the value of a_0^* from the parameters of the static problem, we used Eq.4.

For an initially horizontal ribbon, using $Y_\beta/S_\beta = \int_0^1 \sin(\theta)d\chi$, a first integration of Eq.4 leads to $Y_\beta/S_\beta = \frac{b}{2}[(S_\beta/a_0)^2 - \theta'(0)^2]$. Because of the Γ-region, the longitudinal curvature at the point α is given by a_0/ν and $\theta'(0) = \nu S_\beta/a_0$. The height of the free border is then given simply by $Y_\beta = L_g^3/24a_0^2$, where we define the elasto-gravitational length $L_g = (\frac{Eh^3}{g\sigma})^{1/3}$. Y_β increases with the square of the natural curvature until the critical situation at which the curling proceeds.

We report in the upper inset of Fig.3, different shapes we obtain from the numerical solution of Eq.4. We report also in the lower inset of Fig.3, the stability diagram, where $Y_\beta/S_\beta = \frac{b}{2}(1-\nu^2)\theta'(1)^2$ and $\frac{L_g}{a_0} = [12b(1-\nu^2)]^{\frac{1}{3}}\theta'(1)$ are written in terms of the parameters $(b, S_\beta/a_0)$ plotted in Fig.3. For each value of Y_β/S_β, two solutions are found for two different L_g/a_0: one stable (upper solid line) and one unstable (lower dashed line). No more static solutions are found when $L_g/a_0^* \gtrsim 3.57$.

The critical natural radius a_0^* varies slowly with ν. In the range, $0.3 < \nu < 0.5$, a_0^* varies less than 1%. Thus, curling occurs, in general, only when $a_0 \lesssim 0.28L_g = a_0^*$. For PVC and PP ribbons, we find a_0^* equal to 3.8 ± 0.2 cm and 3.7 ± 0.2 cm respectively, in good agreement with our observations.